Lecture 21. Change of basis

Def Given a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ of $\mathbb{R}^n$, the <u>$\mathcal{B}$-matrix</u> of a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the matrix with columns given by $[T(\vec{v}_1)]_{\mathcal{B}}, [T(\vec{v}_2)]_{\mathcal{B}}, \cdots, [T(\vec{v}_n)]_{\mathcal{B}}$.

Note (1) If $\mathcal{B}$ is the standard basis of $\mathbb{R}^n$, the $\mathcal{B}$-matrix of T is simply the standard matrix of T.

(2) For a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $m \neq n$, we can similarly define the matrix of T relative to two bases, one for $\mathbb{R}^n$ and the other for $\mathbb{R}^m$.

Thm (Change of basis for linear transformations)

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$. Given a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with standard matrix A and $\mathcal{B}$-matrix $A_{\mathcal{B}}$, we have

$$\boxed{A = BA_{\mathcal{B}}B^{-1}}$$

where B is the matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] \arrow[d, "B^{-1}"'] & \mathbb{R}^n & \text{T via standard basis} \\
\mathbb{R}^n \arrow[r, "A_{\mathcal{B}}"] & \mathbb{R}^n \arrow[u, "B"'] & \text{T via basis } \mathcal{B}
\end{tikzcd}$$

Note (1) B must be invertible as its columns form a basis of $\mathbb{R}^n$.

(B is a square matrix with $\det(B) \neq 0$)

(2) We can often find a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix $A_{\mathcal{B}}$ is relatively easy to compute. Then we can use the theorem to compute the standard matrix A.

Ex Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which swaps the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Sol Take $\mathcal{B}$ to be the basis of $\mathbb{R}^2$ given by $\vec{v}_1$ and $\vec{v}_2$.

T swaps $\vec{v}_1$ and $\vec{v}_2$ $\Rightarrow$ $T(\vec{v}_1) = \vec{v}_2$ and $T(\vec{v}_2) = \vec{v}_1$.

$$\Rightarrow [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$(T(\vec{v}_1) = 0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$ and $T(\vec{v}_2) = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2)$

The $\mathcal{B}$-matrix of T is

$$A_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we take the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

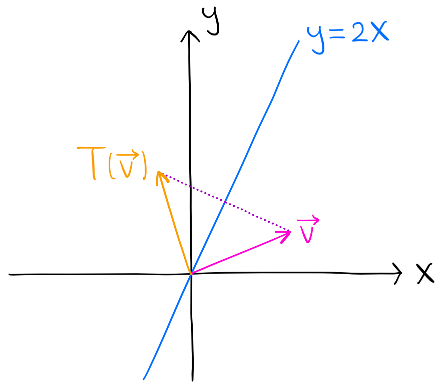
with columns $\vec{v}_1$ and $\vec{v}_2$.

$$\Rightarrow B^{-1} = \frac{1}{2 \cdot 2 - 3 \cdot 1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Hence the standard matrix is

$$A = BA_{\mathcal{B}}B^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -4 & 3 \\ -5 & 4 \end{bmatrix}}$$

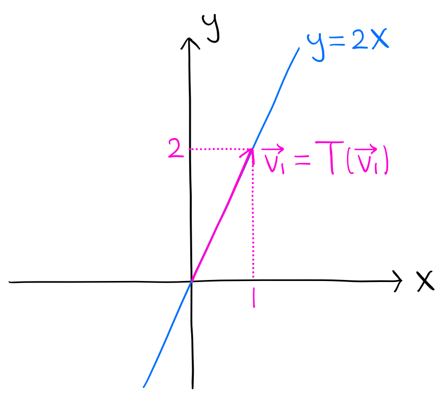
Ex Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which reflects each vector through the line $y = 2x$.

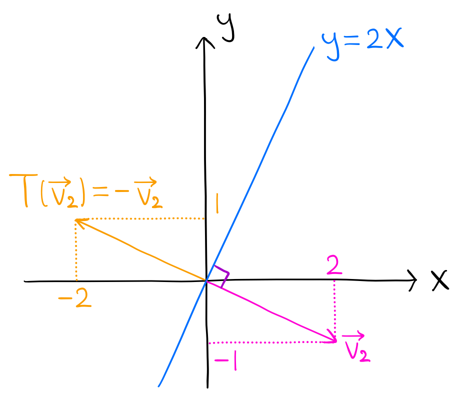


Sol Take $\mathcal{B}$ to be the basis of $\mathbb{R}^2$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

While it is difficult to directly compute $T(\vec{e}_1)$ and $T(\vec{e}_2)$, it is relatively easy to compute $T(\vec{v}_1)$ and $T(\vec{v}_2)$.





$\vec{v}_1$ is on the line $y = 2x \implies T(\vec{v}_1) = \vec{v}_1$

$\vec{v}_2$ is perpendicular to the line $y = 2x \implies T(\vec{v}_2) = -\vec{v}_2$.

($\vec{v}_2$ has slope $-\frac{1}{2}$ while the line $y = 2x$ has slope 2)

Hence we obtain the $\mathcal{B}$-coordinate vectors

$$[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$\left(T(\vec{v}_1) = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \text{ and } T(\vec{v}_2) = 0 \cdot \vec{v}_1 + (-1) \cdot \vec{v}_2 \right)$

The $\mathcal{B}$-matrix of T is

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now we take the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

with columns $\vec{v}_1$ and $\vec{v}_2$.

$$\Rightarrow B^{-1} = \frac{1}{1\cdot(-1) - 2\cdot 2} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Hence the standard matrix is

$$A = BA_{\mathcal{B}}B^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \boxed{\frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}}$$

Note (1) For a reflection through the line $y = mx$ in $\mathbb{R}^2$, we can apply the same strategy with a basis given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} m \\ -1 \end{bmatrix}.$$

(2) We will revisit this example in Lecture 32 and Lecture 34 where we will discuss a general formula for reflections by refining the strategy presented here.